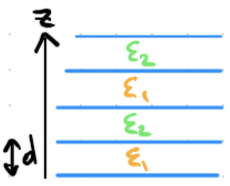
4) Uniaxial materials

An example where the power flow and the wavenumber are not collinear and negative refraction is possible.

1. Description and permittivity tensor



$d \ll \lambda_1, \lambda_2$

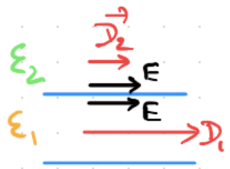
The medium is described by a permittivity tensor

$$\boxed{\hat{\varepsilon} = \varepsilon_0 \begin{pmatrix} \varepsilon_T & 0 & 0 \\ 0 & \varepsilon_T & 0 \\ 0 & 0 & \varepsilon_{zz} \end{pmatrix} \text{ with } \begin{cases} \varepsilon_T = \frac{1}{2}(\varepsilon_1 + \varepsilon_2) \\ \varepsilon_{zz} = \dfrac{2\varepsilon_1\varepsilon_2}{\varepsilon_1 + \varepsilon_2} \end{cases}}$$

$\hat{\mu} = \mu_0 \mathbb{1}_3 \quad , \hat{\alpha} = \hat{\beta} = 0.$

<u>Proof</u>:

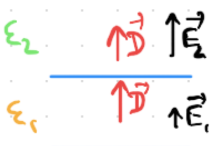
$\boxed{\text{Case } \vec{E} \parallel \vec{x} \text{ or } \vec{y}}$



$\vec{E}$ is the same in each layer by BC $\rightarrow \vec{D}_{eff} = \frac{1}{2}(\vec{D}_1 + \vec{D}_2) \parallel \vec{x}$

$\vec{D}_{eff} = \frac{1}{2}(\varepsilon_1 \vec{E} + \varepsilon_2 \vec{E})\varepsilon_0 = \varepsilon_0 \frac{\varepsilon_1 + \varepsilon_2}{2} \vec{E}$

$\boxed{\text{Case } \vec{E} \parallel \vec{z}}$



$\vec{D}$ is the same in each layer by BC $\Rightarrow \vec{E}_{eff} = \frac{1}{2}(\vec{E}_1 + \vec{E}_2) \parallel \vec{z}$

$\Rightarrow \vec{E}_{eff} = \frac{1}{2\varepsilon_0}\left(\frac{1}{\varepsilon_1} + \frac{1}{\varepsilon_2}\right)\vec{D} \Rightarrow \vec{D} = \left(\frac{\varepsilon_1^{-1} + \varepsilon_2^{-1}}{2\varepsilon_0}\right)^{-1} \vec{E}_{eff} = \varepsilon_{zz}$

Studies of propagation along $\vec{x}, \vec{y}, \vec{z}$ are trivial $\Rightarrow$ let's assume $\vec{k} = k_x \vec{x} + k_z \vec{z}$

2. Eigenproblem

$\hat{M}\vec{E} = \vec{0}$ with $M = \hat{K}^2 + k_0^2 \hat{\varepsilon}$

$$\begin{pmatrix} k_0^2\varepsilon_T - k_z^2 & 0 & k_x k_z \\ 0 & k_0^2\varepsilon_T - k_x^2 - k_z^2 & 0 \\ k_x k_z & 0 & k_0^2 \varepsilon_{zz} - k_x^2 \end{pmatrix} \vec{E} = \vec{0}$$

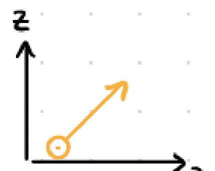
$$\Rightarrow \det M = 0 \Rightarrow 0 = (k_0^2\varepsilon_T - k_z^2)(k_0^2\varepsilon_T - k_x^2 - k_z^2)(k_0^2\varepsilon_{zz} - k_x^2) - k_x^2 k_z^2 (k_0^2\varepsilon_T - k_x^2 - k_z^2)$$

$$\Rightarrow (k_0^2\varepsilon_T - k_x^2 - k_z^2)\left[\underbrace{(k_0^2\varepsilon_T - k_z^2)(k_0^2\varepsilon_{zz} - k_x^2)}_{k_0^2(k_0^2\varepsilon_T\varepsilon_{zz} - \varepsilon_T k_x^2 - \varepsilon_{zz} k_z^2) + \cancel{k_x^2 k_z^2}} - \cancel{k_x^2 k_z^2}\right] = 0$$

$$\Rightarrow (\underbrace{k_0^2\varepsilon_T - k_x^2 - k_z^2}_{\text{sol 1}})\,(\underbrace{k_0^2\varepsilon_T - \frac{\varepsilon_T}{\varepsilon_{zz}} k_z^2 - k_z^2}_{\text{sol 2}}) = 0 \qquad (\varepsilon_{zz} \neq 0)$$

sol 1: $\boxed{k_x^2 + k_z^2 = k_0^2 \varepsilon_T}$

$$\begin{pmatrix} a & 0 & k_x k_z \\ 0 & 0 & 0 \\ k_x k_z & 0 & b \end{pmatrix} \begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix} = 0 \Rightarrow \begin{pmatrix} 0 \\ E_y \\ 0 \end{pmatrix}$$



TE^z polarisation

sol 2 $\quad k_z^2 = k_0^2\varepsilon_T - k_x^2 \dfrac{\varepsilon_T}{\varepsilon_{zz}}$ or $\boxed{\dfrac{k_z^2}{\varepsilon_T} + \dfrac{k_x^2}{\varepsilon_{zz}} = k_0^2}$

$$\begin{pmatrix} k_x^2 \frac{\varepsilon_T}{\varepsilon_{zz}} & 0 & k_x k_z \\ 0 & k_x^2 \frac{\varepsilon_T}{\varepsilon_z} & 0 \\ k_x k_z & 0 & k_0^2 \varepsilon_z - k_x^2 \end{pmatrix} \begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix} = 0$$

If $k_x \neq 0 \Rightarrow E_y = 0$

$$k_x^2 \frac{\varepsilon_T}{\varepsilon_{zz}} E_x = -k_x k_z E_z \Rightarrow \vec{E} = \begin{pmatrix} E_x \\ 0 \\ -\frac{k_y}{k_z} \frac{\varepsilon_T}{\varepsilon_{zz}} E_x \end{pmatrix}, \quad \vec{H} = \frac{1}{\omega\mu_0} \vec{k} \times \vec{E} \sim \vec{y}$$

TM^z polarisation

Magnetic field

$$-j\vec{k} \times \vec{H} = j\omega \hat{\varepsilon} \vec{E} \Rightarrow \vec{k} \times \vec{H} = -\omega \hat{\varepsilon} \vec{E}$$

$$\begin{pmatrix} k_x \\ 0 \\ k_z \end{pmatrix} \times \begin{pmatrix} 0 \\ H_y \\ 0 \end{pmatrix} = -\omega \begin{pmatrix} \varepsilon_T & 0 & 0 \\ 0 & \varepsilon_T & 0 \\ 0 & 0 & \varepsilon_{zz} \end{pmatrix} \begin{bmatrix} E_x \\ E_y \\ E_z \end{bmatrix} \Rightarrow \begin{cases} H_y = \dfrac{\omega \varepsilon_T}{k_z} E_x \\ H_y = -\dfrac{\omega \varepsilon_{zz}}{k_x} E_z \end{cases}$$

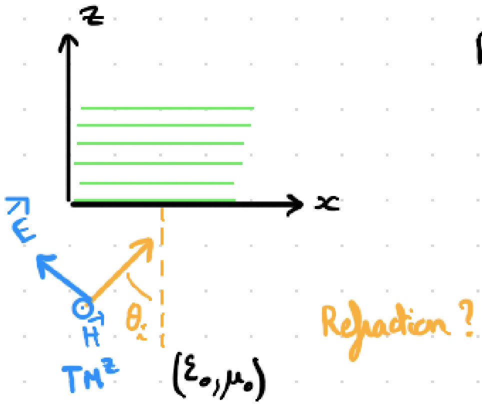
Poynting vector

$$\vec{S} = \frac{1}{2} \vec{E} \times \vec{H}^* = \frac{1}{2} \begin{pmatrix} E_x \\ 0 \\ E_z \end{pmatrix} \times \begin{pmatrix} 0 \\ H_y^* \\ 0 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} -E_z H_y^* \\ 0 \\ H_y^* E_x \end{pmatrix} = \frac{1}{2} \begin{pmatrix} \dfrac{k_x}{\omega \varepsilon_{zz}} |H_y|^2 \\ 0 \\ \dfrac{k_z}{\omega \varepsilon_T} |H_y|^2 \end{pmatrix}$$

$\vec{S}$ is not parallel to $\vec{k}$: $\vec{E} \perp \vec{S}$ but $\vec{E} \not\perp \vec{k}$

If layers are metal & dielectric $\varepsilon_\wedge > 0$, $\varepsilon_z < 0$ $\vec{S}$ & $\vec{k}$ can differ strongly in direction.

3. All angle negative refraction



Reminder :
$$\begin{cases} \dfrac{k_x^2}{\varepsilon_{zz}} + \dfrac{k_z^2}{\varepsilon_T} = k_0^2 \\ S_z = \dfrac{k_z}{2\omega\varepsilon_T} |H_y|^2 \quad \text{for } TM^z \text{ pol} \\ S_x = \dfrac{k_x}{2\omega\varepsilon_{zz}} |H_y|^2 \end{cases}$$

Refraction angle in terms of power

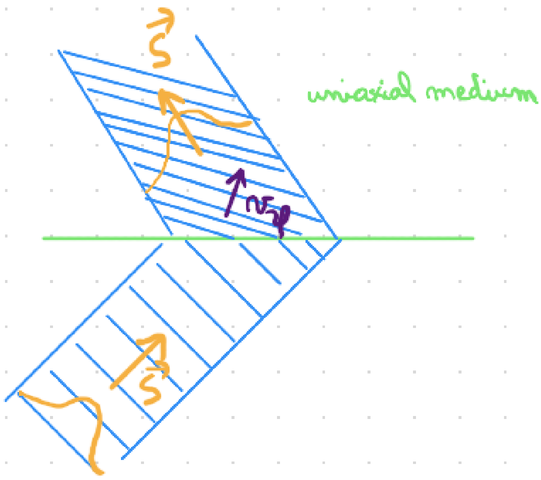
continuity of k_x

$$\theta_r^{(S)} = \arctan \frac{S_x}{S_z} = \arctan\left(\frac{k_x}{k_z}\frac{\varepsilon_T}{\varepsilon_{zz}}\right) = \arctan\left(\frac{\varepsilon_T}{\varepsilon_{zz}}\frac{k_0 \sin\theta_i}{k_z}\right)$$

Refraction angle for phase

$$\theta_r = \arctan\left(\frac{k_0 \sin\theta_i}{k_z}\right)$$

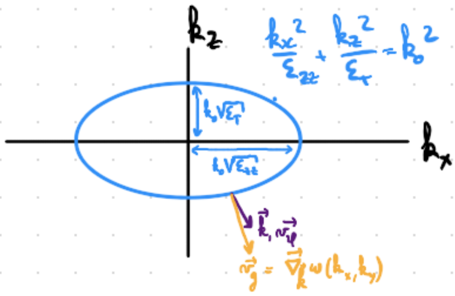
If $\dfrac{\varepsilon_T}{\varepsilon_{zz}} < 0 \Rightarrow$ both angles have opposite sign $\Rightarrow$ $\boxed{\text{negative refraction angle } \forall \theta_i}$



Message : can only be visualized using a Gaussian beam instead of plane waves

4. Hyperbolic dispersion

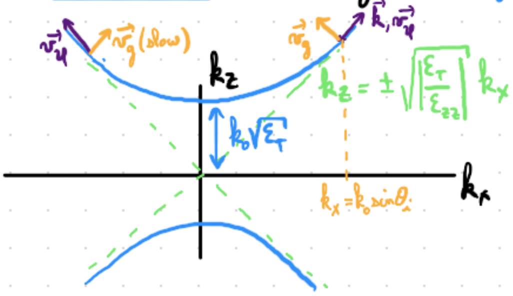
$\underline{\varepsilon_T, \varepsilon_{zz} > 0}$, $\varepsilon_T \neq \varepsilon_z$ $\rightarrow$ elliptic



$\Rightarrow$ still a cut off

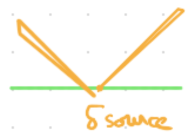
$\Rightarrow$ diffraction limited

$\underline{\varepsilon_T / \varepsilon_z < 0}$ $\rightarrow$ hyperbolic [eg $\varepsilon_T > 0, \varepsilon_{zz} < 0$, $\frac{k_z^2}{\varepsilon_T} - \frac{k_x^2}{|\varepsilon_{zz}|} = k_0^2$]



$\Rightarrow$ no cut-off

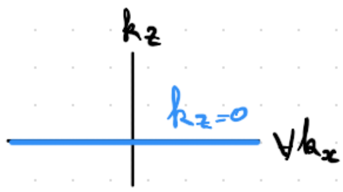
$\Rightarrow$ details are transmitted



conical radiation with small details transmitted

Interesting case : $\varepsilon_1 = -\varepsilon_2$ $\Rightarrow$ $\varepsilon_T \rightarrow 0$ $\quad \varepsilon_{zz} \rightarrow +\infty$

$\Rightarrow k_z \rightarrow 0 \quad \forall k_x$



$$S_z = \frac{k_z |H_y|^2}{2\omega\varepsilon_T} \text{ undetermined}$$

$$S_x = \frac{k_x}{2\omega\varepsilon_z}|H_y|^2 \rightarrow 0$$

$\rightarrow$ you see the source in the far field with no phase shift and no diffraction = canalization

Rq: To transmit power efficiently you need $n\frac{\lambda}{2}$ in thickness (Fabry-Pérot) $\rightarrow \infty$ material. A more practical way is to have different thicknesses:

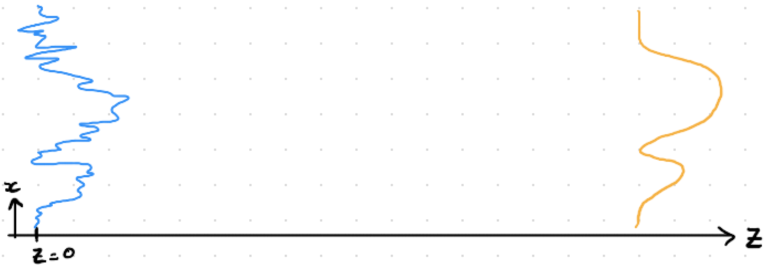
$$\varepsilon_T = \frac{d_1\varepsilon_1 + d_2\varepsilon_2}{d_1 + d_2} \not\rightarrow 0 \qquad \varepsilon_{zz} = \frac{(d_1+d_2)\,\varepsilon_1\varepsilon_2}{d_1\varepsilon_2 + d_2\varepsilon_1} \; \begin{matrix}\longrightarrow +\infty \\ \rightarrow 0\end{matrix}$$

We still have $S_x \rightarrow 0$, S_z finite

5. Abbe’s limit and a Hyperlens

Abbe's diffraction limit

$\boxed{\text{Field variation details smaller than } \lambda \text{ are lost in the far field}}$



$$E_y(x, z=0) = \underbrace{\int_{-\infty}^{+\infty} \tilde{E}_y(k_x)\, e^{-jk_x x}\, dk_x}_{\text{large } k_x \to \text{ fine details}} \quad \text{with } \tilde{E}_y(k_x) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} E_y(x) e^{jk_x x}$$

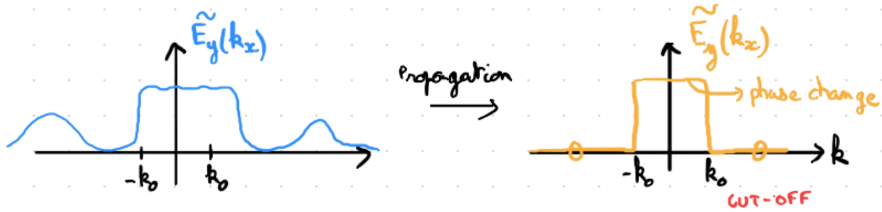
At $z > 0$:

$$E_y(x, z) = \int_{-\infty}^{+\infty} \tilde{E}_y(k_x)\, e^{-jk_x x}\, e^{-j k_z z} \qquad \text{with } k_x^2 + k_z^2 = k_0^2$$

$$\begin{cases} k_x < k_0 \Rightarrow k_z = \sqrt{k_0^2 - k_x^2} \in \mathbb{R} \rightarrow \text{propagates to the FF} \\ k_x > k_0 \Rightarrow k_z = -j\sqrt{k_x^2 - k_0^2} \in -j\mathbb{R}^+ \rightarrow \text{evanescent} \rightarrow 0 \text{ in FF} \end{cases}$$

$$k_x^{max} = \frac{2\pi}{\Delta x} = k_0 \Rightarrow \underline{\Delta x = \lambda}$$

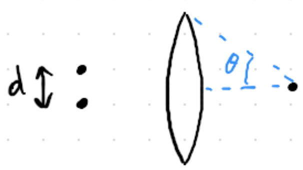
resolution



A lens is a far field device that compensates the phase change. However, in practice, the lens has a final numerical aperture ⇒ only a subset of $[-k_0, k_0]$ is compensated.

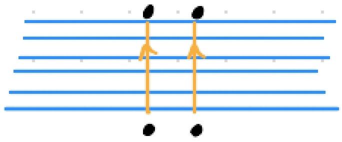
⇒ the real limit is about distinguishing 2 point source, spaced by d, with a finite NA:

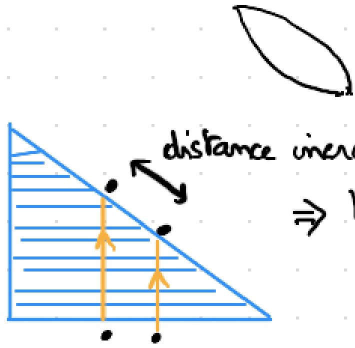
$$\boxed{d = \frac{\lambda}{2\,NA}} \quad \text{Abbe's diffraction limit} \quad (NA = n \sin\theta)$$



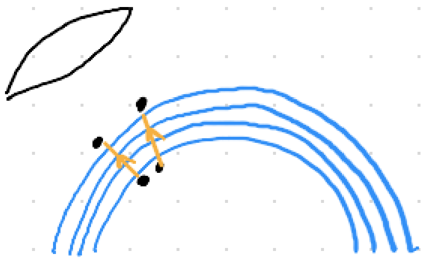
$d < \frac{\lambda}{2} \Rightarrow$ single point in FF

The hyperlens



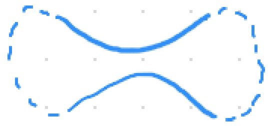


distance increased by angle
⇒ the 2 sources can be picked up by a regular lens



cylindrical version
$\frac{\lambda}{10}$ demonstrated

but: problem of losses that close the hyperbolic contours.



Conclusion : Subwavelength imaging requires:

- access to the near field to pick up details (hyperlens, NSOM, see also perfect lens by J. Pendry)

OR - multiple illuminations (2014 Chemistry Nobel Prize)

(Fluorescent) Photoactivable localization microscopy PALM, FPALM
Stochastic optical reconstruction microscopy STORM